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A Class of Analytic Interference Models*

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We present a class of interference models with good analytic properties, arbitrary trajectory functions, no ancestors or ghosts, and a fair amount of freedom remaining. Asymptotics are good for bounded trajectories; for linearly rising trajectories there is trouble in one direction. As an example we do a small calculation on $X^0 \to \eta\pi\pi$.

I. INTRODUCTION

In the current state of the art of high-energy theory, many problems come down to constructing a model amplitude after one has extracted the known kinematic, spin and internal-symmetry factors. One tries to put in as many of the basics as possible – Lorentz invariance, analyticity, crossing symmetry, unitarity, Regge asymptotics – and one always has to violate one or more of these. This leads to various compromises, where one builds in some features and either ignores others (hopefully only temporarily) or tries to satisfy the others in some approximate sense.

One can also have a general scheme – e.g., Regge-cut machines,[1,2] eikonal generators,[2,3] multi-Regge equations[4,5] – in which one uses a basic model amplitude as an input to obtain a "better" output. Ultimately, of course, one wants to do a comparison with the experimental data, but this is usually done by starting from a model with good general properties and a small number of arbitrary parameters, and then juggling parameters to obtain the best possible fit to the data.

There are almost as many theoretical models as there are theorists, but in the last few years the bulk of the models has concentrated on crossing symmetry, duality, and narrow-resonance approximations.[6] One of the more critical diseases of these models is their difficulty in moving away from the narrow-resonance approximation. If one tries to use a complex trajectory function in order that resonances may have nonzero widths, one runs into the problem of "ancestors" – the resonance is predicted to occur in *all* partial waves at the same energy. It was in looking for ways around this difficulty that we came up with the model class to be described in detail in Sec. II. Other workers[7] have also found solutions to the ancestor problem, usually in the context of dual models. They have trouble with asymptotics, as we do also; they also have what are to our minds complicated and restrictive representations.

Basically, our models can be taken as pieces of an interference model, possibly symmetrized for crossing, or as input in one of the model-modifying machines, or directly in the same way that Regge models are used for phenomenology. The good features are that an arbitrary trajectory function can be used with no resulting ancestors (nor daughters, nor ghosts if desired), analytic properties are good, asymptotics are good almost everywhere, and there remains a great deal of freedom left in the functional forms. Bad features include the explicit lack of "atonous duality,"[6] provided one believes that is significant; the nonunitary nature common to most recent models, and the possible trouble with asymptotics in one region. In regard to the last and the first, Jengo[8] has shown that essentially any reasonable amplitude can be written in an interference form: The total amplitude is a sum of individual amplitudes, each of which has poles in one variable, Regge asymptotics in another, and goes to zero faster than any power in all directions other than the one which has Regge asymptotics. Clearly, our amplitude class is not yet of this type, but we believe it is interesting enough in its own right to be worth considering.

In Sec. III we do a small calculation on X^0 decay and compare with Moffat's[9] model and the data.

II. THE MODELS

We wish to consider models of the type

$$F(\alpha, x)=\int_0^1 \frac{\zeta^{-\alpha-1}f(\alpha,\zeta)d\zeta}{(1-x\zeta)^{\beta}}\,, \tag{1}$$

where we have in mind the identification $\alpha=\alpha(s)$ or $\alpha(t)$ and $x=s/s_0$ or t/t_0, and where s_0 (t_0) are the appropriate thresholds. This form is only defined for $\mathrm{Re}\,\alpha<0$ and must be continued to other values. $f(\alpha,\zeta)$ is analytic at $\zeta=0$ and entire in α; β is arbitrary and can be chosen to suit one's convenience. A full amplitude would be a combination of forms of the type of Eq. (1).

The analytic properties are straightforward; there is a cut in x from 1 to ∞, and if α is bounded and satisfies a dispersion relation then F has a Mandelstam representation. Because of the good analytic properties, satisfaction of analyticity-derived relations such as finite-energy sum rules will be automatic if asymptotics are correct. There are poles in α at the integers, whose residues are determined by the derivatives of $f(\alpha,\zeta)$ at the origin. Let, for $|\zeta|<\epsilon$, $f(\alpha,s)$ have the convergent representation

$$f(\alpha,\zeta)=\sum_{n=0}^{\infty}\frac{f_n(\alpha)\zeta^n}{\Gamma(n+1)}\,. \tag{2}$$

Then, the poles in α come only from the neighborhood of $\zeta=0$, and we can legitimately use Eq. (2) in Eq. (1), throwing away the integral from ϵ to 1 to evaluate (by continuing from $\mathrm{Re}\,\alpha<0$ to $\mathrm{Re}\,\alpha>0$)

$$\lim_{\alpha\to N}(N-\alpha)F(\alpha,\zeta)=\sum_{\nu=0}^{N}\frac{f_{N-\nu}(N)\Gamma(\beta+\nu)x^{\nu}}{\Gamma(\nu+1)\Gamma(N-\nu+1)\Gamma(\beta)}\,. \tag{3}$$

We see immediately that the residue at $\alpha=N$ is a polynomial of degree N in x, that it depends on only the first N partial derivatives of f with respect to ζ evaluated at $\alpha=N$, and that if we choose these derivatives correctly we can obtain any polynomial of degree N. Thus we automatically have no ancestors, and we can choose the residues to be zero or positive quantities times Legendre polynomials[10] in $\cos\theta$, thus specifying all parent and daughter widths and eliminating ghosts. Of course, for a given specification one has to check back and see that $f(\alpha,\zeta)$ will actually exist.

As an example, let us take the case of equal mass ($m=1$) scattering, so that $x=t/4$ and $\cos\theta \equiv z=1+2t/(s-4)$. If we want the resonance at $\alpha(s_N)=N$ (on the second sheet) to be just in the Nth partial wave, i.e., no daughters, then we want the residue to be

$$\begin{aligned}C_N P_N(z)&\equiv C_N F(-N,N+1,1,(1-z)/z)\\ &=C_N\sum_{\nu=0}^{N}\frac{\Gamma(N+1+\nu)}{\Gamma(N+1-\nu)\Gamma(\nu+1)^2}\left(\frac{t}{s_N-4}\right)^{\nu}.\end{aligned} \tag{4}$$

Comparing Eq. (3) and Eq. (4) we see that we want

$$f_n(N)=C_N\frac{\Gamma(\beta)\Gamma(2N+1-n)}{\Gamma(\beta+N-n)\Gamma(N+1-n)}\left(\frac{4}{s_N-4}\right)^{N-n},\qquad n=0,1,2,\ldots,N. \tag{5}$$

If we take this form for all n, this implies

$$f(N,\zeta)=\frac{C_N(\nu_N)^{-N}\Gamma(1+2N)\Gamma(\beta)}{\Gamma(N+1)\Gamma(N+\beta)}\times F(-N,1-\beta-N,-2N,-\nu_N\zeta)\,, \tag{6}$$

where $\nu=(s-4)/4$ and F is the usual hypergeometric function. We now want to generalize Eq. (6) to arbitrary α; the simplest way is just to replace N by α and ν_N by ν.[11] However, this will result in a form with unwanted poles in α, and a cut in s going down from 4. A form which has good α-dependence and reduces to Eq. (6) at positive integers is

$$f(\alpha,\zeta)=\frac{C(\alpha)\Gamma(\beta)\sqrt{\pi}\,(-\nu/4)^{-\alpha}}{\Gamma(\alpha+\beta)\Gamma(\frac{1}{2}-\alpha)}\times F(-\alpha,1-\beta-\alpha,-2\alpha,-\nu\zeta)\,. \tag{7}$$

Lest this look terribly exotic, we should mention that for $\beta=\frac{1}{2}$ we have

$$f(\alpha,\zeta)=C(\alpha)\cos\pi\alpha\,(-\nu/4)^{-\alpha}\times\frac{1}{(1+\nu\zeta)^{1/2}}\left(\frac{1+(1+\nu\zeta)^{1/2}}{2}\right)^{2\alpha+1}$$

and $F(\alpha,x)$ is just the old strip-approximation function,[12] which was constructed precisely for its nice analytic properties.[13]

Returning to a consideration of Eq. (3), since at $\alpha=N$ we only need to specify the first N derivatives of f, we are free to make the replacement

$$f_n(\alpha)\to f_n(\alpha)+g_n(\alpha)/\Gamma(n-\alpha)\,.$$

When this second term is substituted back into Eq. (1), we obtain

$$G(\alpha,x)=\int_0^1 d\zeta\,\frac{g(\alpha,\zeta)F(1,\beta,-\alpha,(1-\zeta)x/(x-1))}{(1-\zeta)^{\alpha+1}(1-x)^{\beta}\Gamma(-\alpha)}\,, \tag{8}$$

where $g(\alpha,\zeta)$ is expanded as in Eq. (2). The function $G(\alpha,x)$ has no poles in α, as the behavior of the integrand near $\zeta=1$ is in fact regular as α goes to an integer. The function $g(\alpha,\zeta)$ need only be analytic in a neighborhood of $\zeta=0$ and entire in α; it is otherwise arbitrary.

Let us now turn our attention to the x behavior in Eq. (1). There is a cut in x from 1 to ∞, and by continuing the representation to values of x greater than 1, we find the discontinuity as

$$F(x+i\epsilon)-F(x-i\epsilon)=2i\sin\pi\beta\int_{1/x}^{1}\frac{\zeta^{-\alpha-1}f(\alpha,\zeta)}{(x\zeta-1)^{\beta}}\,d\zeta\,.$$

In order to find the behavior of the discontinuity as $x\to1$ [which should be as $(x-1)^{1/2}$ for a unitarized phase shift], we need to know what $f(\alpha,\zeta)$ looks like near 1. If $f(\alpha,\zeta)\sim(1-\zeta)^{\gamma}$ as $\zeta\to1$, then the discontinuity $\sim(x-1)^{\gamma-\beta+1}$. In our example in Sec. III we shall choose $\gamma=\beta-\frac{1}{2}$.

For the asymptotics in x, we first notice that for $\zeta>\epsilon>0$ in the integration in Eq. (1), as $|x|\to\infty$ this piece of $F(\alpha,x)$ will behave like $|x|^{-\beta}$. In fact, if $\mathrm{Re}(\alpha+\beta)<0$, F itself will behave this way. However, for $\mathrm{Re}(\alpha+\beta)>0$, which we will henceforth assume, the behavior of the integrand near $\zeta=0$, which also controls the poles in α, will control the asymptotic behavior in x at fixed α. This is, of course, not unexpected, since this is precisely the usual Regge-pole connection. Roughly speaking, only the region $\zeta\sim1/x$ becomes significant.

The result is that $|F|\sim|x|^{\alpha}$ as $|x|\to\infty$. In particular, as $x\to\infty+i\epsilon$ with α fixed and $(\alpha+\beta)>0$

$$F(\alpha,x)\to\frac{x^{\alpha}f(\alpha,0)\Gamma(\beta+\alpha)}{\Gamma(\beta)\Gamma(1+\alpha)}\,\pi(i-\cot\pi\alpha)\,. \tag{9}$$

This is, including the phase factor, the usual Regge result.

The asymptotic behavior in the other variable is much more complicated, and depends on the specific form chosen. Of course, if $\alpha(s)$ is bounded, then the asymptotics are easy, since we need only continue F to the value for $\alpha(\infty)$. However, in the currently favored[14] case of infinitely rising trajectories we must find the continuations to the whole α plane. Generally speaking, for $\alpha\to-\infty$ the form of Eq. (1) should be suitable, and suggests a power-law decrease. Specifically, if $f(\alpha,\zeta)\to\tilde{f}(\alpha)(1-\zeta)^{\gamma}$ as $\zeta\to1$, and further $f(\alpha,\zeta)$ is well behaved in ζ for all α, then because of the factor $\zeta^{-\alpha-1}$ only the region near $\zeta=1$ will contribute to the leading asymptotics and

$$F(\alpha,x)\to(-\alpha)^{-\gamma-1}(1-x)^{-\beta}\Gamma(\gamma+1)\tilde{f}(\alpha)$$

as $\alpha\to-\infty$, x fixed. For $\alpha\to+\infty$, in order to do the continuation one can write the integration in Eq. (1) as a contour looping around the origin where one picks up a phase contribution $e^{-2\pi i\alpha}$ from one side; then open up the contour, using $\alpha\to+\infty$ to eliminate contributions from the infinite semicircles; and have remaining integrations over the other cuts. These integrations will usually be suitable for the limit $\alpha\to+\infty$. Our experience seems to indicate that the limit which will give the most trouble is $\alpha\to+\infty$ simultaneously with $x\to-\infty$.

Two other brief comments before we go on to our example: First, as is clear from the asymptotics for fixed α and large x, when x is the momentum-transfer variable $F(\alpha,x)$ will in general have Regge poles at $\alpha,\alpha-1,\ldots,$ and fixed poles at $-\beta,-\beta-1$. Second, if one tries to construct a $f(\alpha,\zeta)$ such that F is symmetric, i.e.,

$$F(\alpha(y),x)=F(\alpha(x),y)\,,$$

then as $x\to s_N$ so that $\alpha(x)=N$, there is a pole in x. When we continue $F(\alpha(y),x)$ to the same point, it must also have a pole, and hence $f(\alpha,\zeta)$ must have a branch point, independent of the value of α, at $\zeta=1/s_N$. If α is a rising trajectory, so that $s_N\to\infty$ (as $N\to\infty$), this means that $f(\alpha,\zeta)$ must have an essential singularity at the origin, in contradiction to our initial assumptions.

III. A SAMPLE CALCULATION

We wish to choose a simple form of the amplitude to describe $X^0\to\eta\pi\pi$ in the same way as Lovelace[15] and subsequently other authors[16] applied the off-mass-shell continuation of $\pi\pi$ scattering to $K\to 3\pi$, $\eta\to\pi^+\pi^-\pi^0$, and $\bar{p}n\to\pi^+\pi^-\pi^-$ in a Veneziano[17] model; and as Moen and Moffat[9] applied continuations of their model of $\pi\pi$ to $X^0\pi$, $\eta\pi$ and thence to $X^0\to\eta\pi\pi$. After completion of this work, we received a report by Sivers[18] which casts doubt on the whole final-state-interaction approach to this problem; so our result should be taken more as a comparison with other models to show that this, too, is a not unreasonable amplitude rather than as an actual data prediction.

The $\pi\pi$ scattering amplitude in all the above studies is written as a linear combination of $F(s,t)$, $F(s,u)$, $F(t,u)$, $F(t,s)$, $F(u,s)$, and $F(u,t)$ with coefficients restricted by isospin and crossing. For comparison, the basic amplitude used by Lovelace was

$$F(s,t)=-\beta\frac{\Gamma(1-\alpha(s))\,\Gamma(1-\alpha(t))}{\Gamma(1-\alpha(s)-\alpha(t))}+\gamma\frac{\Gamma(1-\alpha(s))\,\Gamma(1-\alpha(t))}{\Gamma(2-\alpha(s)-\Gamma(t))} \tag{10}$$

whereas the Moen and Moffat amplitude was

$$F(s,t)=-\gamma e^{-\beta s^2}\frac{\Gamma(1-\alpha(s))\,(\alpha_0+\alpha' t)^{\alpha(s)}}{\Gamma(\frac{1}{2}-\alpha(s))}. \tag{11}$$

In line with our earlier comments on the x discontinuity, we shall choose $f(\alpha,\zeta)=(1-\zeta)^{\beta-1/2}$ in order to have a modicum of unitarity. This is about the simplest form, and clearly it could be modified extensively, for example by multiplying $f(\alpha,\zeta)$ by a polynomial in ζ with arbitrary coefficients [this is equivalent to adding satellites for a form like Eq. (10)]. Thus, with $\alpha=\alpha(s)$ and $x=t/t_0$,

$$\begin{aligned}F(s,t)&=F(\alpha,x)\\&=\int_0^1 d\zeta\,\frac{\zeta^{-\alpha-1}(1-\zeta)^{\beta-1/2}}{(1-x\zeta)^{\beta}}\\&=\frac{\Gamma(-\alpha)\Gamma(\beta+\frac{1}{2})}{\Gamma(\beta-\alpha+\frac{1}{2})}\,F(\beta,-\alpha,\beta-\alpha+\tfrac{1}{2},x)\,.\end{aligned} \tag{12}$$

The hypergeometric function and its continuations are well known,[19] and we can explicitly evaluate all of the asymptotic limits [the analytic and pole residue properties are evident in Eq. (12)]. One point on which one has to be careful is the evaluation of limits ($\alpha\to\pm\infty$) inside a hypergeometric series, as lack of uniform convergence can make the results misleading. The relevant results are

$$\begin{aligned}&\text{(a) } \alpha \text{ fixed, } |x|\to\infty, && F\sim|x|^{\alpha}\text{ for }\mathrm{Re}(\alpha+\beta)>0\\&&&\quad\sim|x|^{-\beta}\text{ for }\mathrm{Re}(\alpha+\beta)<0,\\&\text{(b) } x \text{ fixed, } \alpha\to-\infty, && \alpha^{-\beta-1/2},\\&\text{(c) } x \text{ fixed, } \alpha\to+\infty, \text{ or } x\to-\infty,\ \alpha\to+\infty, && |x|^{\alpha}\alpha^{\beta-1}+|x|^{-\beta}\alpha^{-\beta-1/2},\\&\text{(d) } x\to+\infty,\ \alpha\to-\infty, && |x|^{-\beta}|\alpha|^{-\beta-1/2}.\end{aligned} \tag{13}$$

The first term gives Regge asymptotics and all other terms give fixed poles except (c). One way to overcome this is to restrict the model to non-infinitely rising trajectories. Another method is to include a convergence factor in $F(s,t)$ to both improve the asymptotic behavior and fit the $X^0\to\eta\pi\pi$ data, although a certain simplicity in the model is lost.

The model is now constrained further to give results for the particular reaction $X^0\to\eta\pi\pi$, and we shall see that it gives a good fit to the data. The pion is continued off the mass shell to the X^0 (or η) mass, allowing β to be possibly a function of m_π. For $X^0\pi$ (or $\eta\pi$) scattering, the amplitude must be s-u symmetric, and only even-signature trajectories contribute. Furthermore, since the A_2 is exchanged in the s and u channels and the f_0 in the t channel, the trajectory used is the ρ trajectory,[9,20]

$$\alpha(s)=0.483+0.885\,s+i\,0.24(s-4m_\pi^{\,2})^{1/2}\,. \tag{14}$$

Imposing s-u symmetry (unequal-mass scattering) gives the amplitude for $X^0\pi$ scattering (and similarly for $\eta\pi\pi$ scattering):

$$A_{X^0\pi}(s,t,u)=b\,\frac{\Gamma(-\alpha(t))\,\Gamma(\beta+\frac{1}{2})}{\Gamma(\beta-\alpha(t)+\frac{1}{2})}\left[F(\beta,-\alpha(t),\beta-\alpha(t)+\tfrac{1}{2},s/s_0)+F(\beta,-\alpha(t),\beta-\alpha(t)+\tfrac{1}{2},u/u_0)\right], \tag{15}$$

where $s_0=u_0=(m_{X^0}+m_\pi)^2$. The parameter β may be chosen to satisfy the Adler[21] condition when one of the pions is taken off the mass shell and its four-momentum tends to zero; b is a constant.

Now, the amplitude for $X^0\to\eta\pi\pi$ is obtained by continuing one X^0 off shell to the η mass in $X^0\pi$ scattering, or by continuing one η off shell to the X^0 mass in $\eta\pi$ scattering. In either case, b, β, s_0, and u_0 may change, but otherwise the amplitude stays essentially the same. Note, however, that the continuation from $X^0\pi$ scattering gives the thresholds $s_0=u_0=(m_{X^0}+m_\pi)^2$ whereas that from $\eta\pi$ scattering gives $s_0=u_0=(m_\eta+m_\pi)^2$. Thus to incorporate both thresholds in the amplitude, the sum of the terms is taken:

$$A_{X^0\to\eta\pi\pi}(s,t,u)=A\frac{\Gamma(-\alpha(t))\,\Gamma(\beta_1+\frac{1}{2})}{\Gamma(\beta_1-\alpha(t)+\frac{1}{2})}\left[F(\beta_1,-\alpha(t),\beta_1-\alpha(t)+\tfrac{1}{2},s/s_{01})+F(\beta_1,-\alpha(t),\beta_1-\alpha(t)+\tfrac{1}{2},u/u_{01})\right]$$
$$+B\frac{\Gamma(-\alpha(t))\,\Gamma(\beta_2+\frac{1}{2})}{\Gamma(\beta_2-\alpha(t)+\frac{1}{2})}\left[F(\beta_2,-\alpha(t),\beta_2-\alpha(t)+\tfrac{1}{2},s/s_{02})+F(\beta_2,-\alpha(t),\beta_2-\alpha(t)+\tfrac{1}{2},u/u_{02})\right], \tag{16}$$

where

$$s_{01}=u_{01}=(m_{X^0}+m_\pi)^2,$$
$$s_{02}=u_{02}=(m_\eta+m_\pi)^2.$$

The absence of a 0^+ resonance in the Dalitz plot[22] at the mass of the ρ imposes no further restriction; it has already been removed. β_1 and β_2 approximately satisfy the Adler condition for $s=m_\eta{}^2$, $u=m_{X^0}{}^2$, $t=m_\pi{}^2$. β_1 is chosen such that the first two terms equal zero; however, the fourth term with u/u_{02} in this region is unphysical, so β_2 is chosen so as to minimize the modulus of the sum of the second two terms. At β_2 the sum is less than one fifth of its minimum value contributing to the Dalitz plot. The justification for breaking the terms up in this manner stems from the fact that the amplitude for $X^0\pi$ scattering, except for different masses, is the first two terms, and there β is chosen to satisfy the Adler condition for those two terms alone. A similar situation holds for $\eta\pi$ scattering. The values used were $\beta_1=0.015$ and $\beta_2=0.009$, calculated for $\alpha=0.51$.

The modulus squared of the amplitude was used to calculate the Dalitz plot for $X^0\to\eta\pi\pi$; A and B were found by fitting the calculated distribution in normalized kinetic-energy T_η coordinate

$$Y=\left(\frac{2m_\pi+m_\eta}{m_\pi Q}\right)T_\eta-1, \tag{17}$$

with the experimental data[22] and with Moen and Moffat[9] (see Fig. 1). The values found were approximately $A=-0.8$ $B=+1.0$. It should be noted that A and B are the only parameters fit by the data; other parameters are determined before this point. Figure 2 shows the angular dependence of the decay as a function of $|\cos\theta|$; θ is the angle between the η momentum in the X^0 center-of-mass

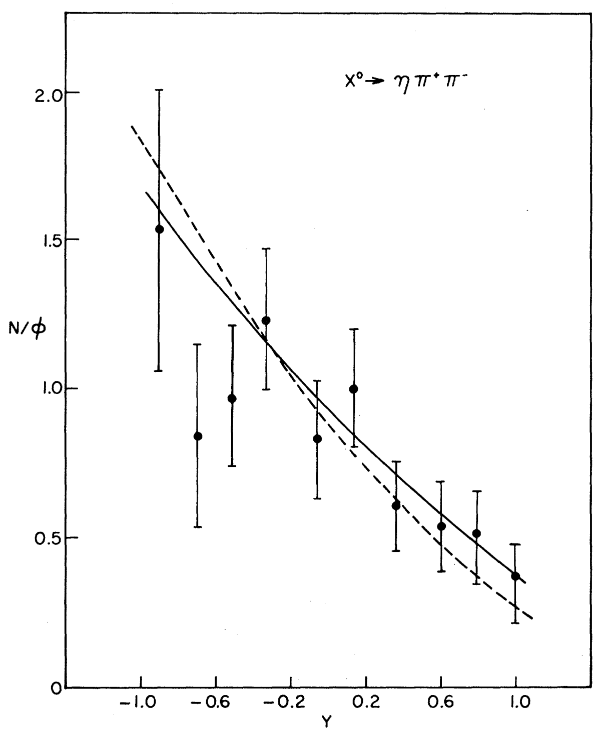

FIG. 1. Distribution divided by phase space in normalized kinetic-energy T_η coordinate

$$Y=\frac{2m_\pi+m_\eta}{m_\pi Q}T_\eta-1.$$

The solid curve represents the present model, the dashed curve, that of Moen and Moffat (Ref. 9). The data are from Ref. 22.

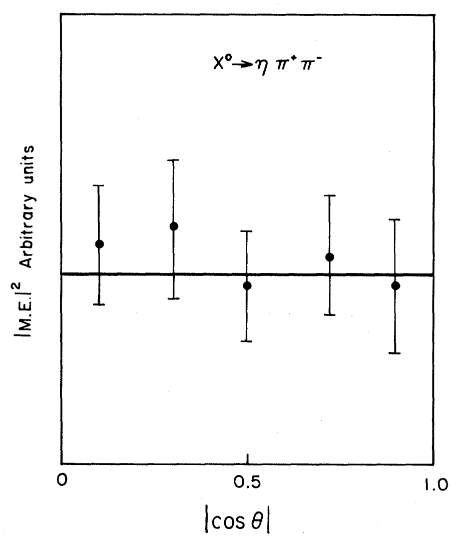

FIG. 2. The angular dependence of the matrix element squared versus $|\cos\theta|$; θ is the angle between the η momentum in the X^0 center-of-mass frame and the π^+ momentum in the dipion rest frame.

frame and the π^+ momentum in the dipion rest frame. As can be seen from the figures, this model gives essentially the same curves as Moffat's model, and we would expect that in other reactions we would also be able to get reasonable results.

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Calculation of the Second-Order Weak Amplitude for $K_2 \to \pi^0 e^+ e^-$*

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The second-order weak amplitude for $K_2 \to \pi^0 e^+ e^-$ has a finite imaginary part arising from the presence of a $\pi\nu$ intermediate state in the channels $K_2 + e^\pm \to \pi^0 + e^\pm$, which we calculate in terms of the amplitudes for $K_2 \to \pi^\pm e^\mp \nu$ and $\pi^\pm \to \pi^0 e^\pm \nu$. The real part is determined by means of a dispersion relation, and the result converges for a simple choice of the K_{2e3} and π_{e3} form factors.

A major unknown in weak-interaction physics is the magnitude and structure of amplitudes that are second-order in the Fermi constant G. In a recent survey of the experimental possibilities in this field,[1] the decay $K_2 \to \pi^0 e^+ e^-$ has been spotlighted as a promising reaction in which to look for such higher-order effects. We describe here a calculation of the second-order weak amplitude for $K_2 \to \pi^0 e^+ e^-$. In the limit in which the effects of strong interaction are ignored, the amplitude diverges logarithmically. Inclusion of the strong-interaction effects in a reasonable way leads to a finite result.

We denote the momenta of the reaction by